Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Test 1
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	6 questions worth 30 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$

$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

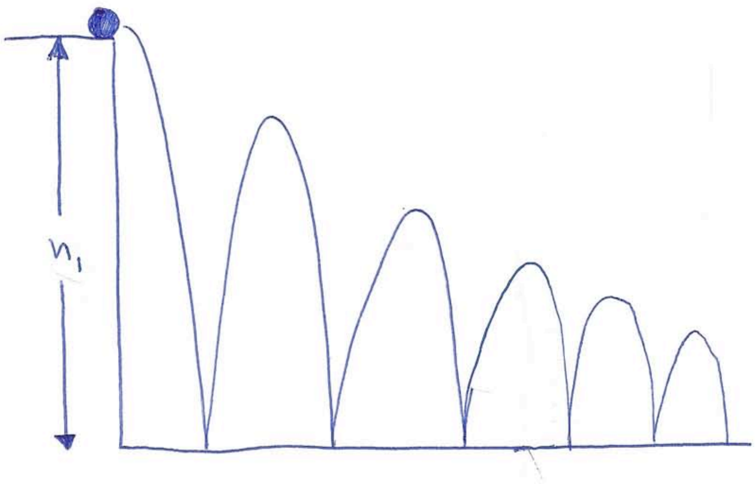
$$\int udv = uv - \int vdu$$

(2 marks each) Determine the 50th number in the sequence and the sum from the first number to the 50th number for each of the following series.

-25, -24, -23, -22, …

1, 1.01, 1.0201, 1.030301, …

(2 marks) If a ball falls from a table at h_1=1m and always rebounds to 97% of the previous bounce. How high will the ball bounce on the 50th rebound.



(2 marks) Take the derivative with respect to “x” of the following function **using the definition of the derivative**.

$$y(x) = ax + b$$

(2 marks each) Take the derivative with respect to “x” of the following functions.

$$y(x) = 2x^4 + sin(x)$$

$$y(x) = 3x^2 sin(x)$$

$$y(x) = 3x^2 sin(3x^2 + 1) + x$$

$$y(x) = \frac{3x^2}{sin(3x^2 + 1)} + x$$

$$y(x) = (x-1)^2 sin((x-1)^3)$$

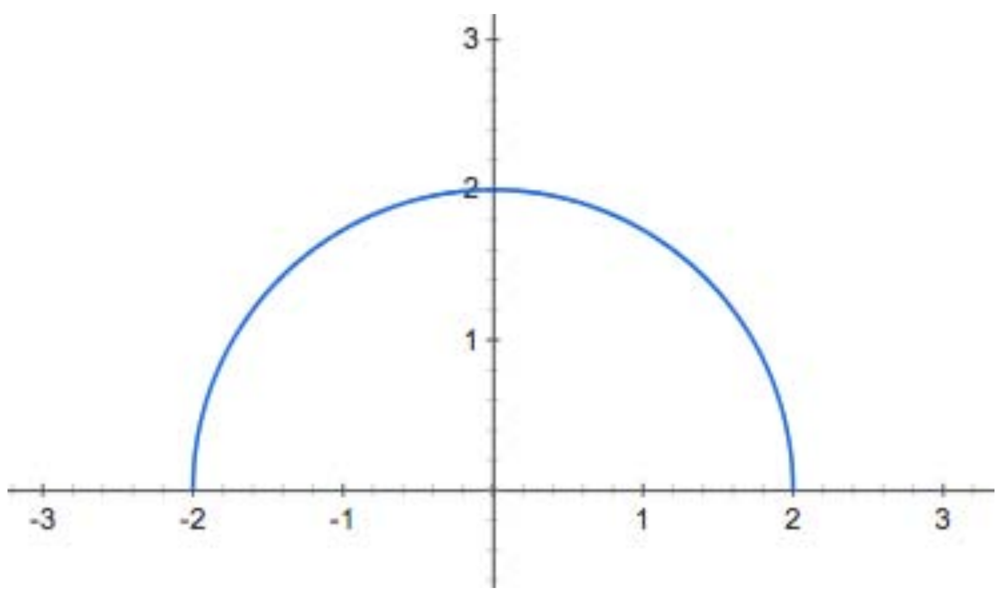
$$y(x) = \left[\left((x-1)^3 - x\right)^3\right]^3$$

(5 marks) Determine the tangent line at x=-1 for the following function. Plot the tangent line on the plot.

$$y(x) = \sqrt{2^2 - x^2}$$

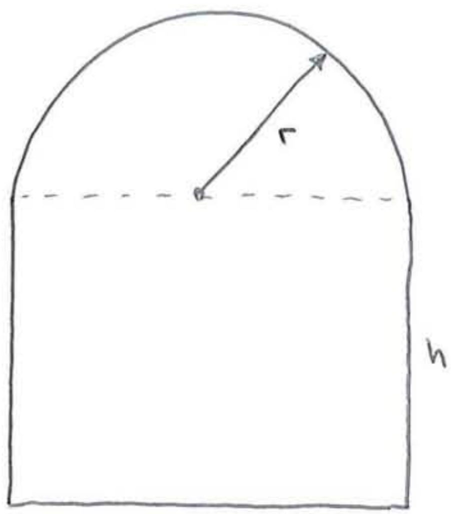
equation of a line:

$$y(x) = ax + b$$



(5 marks) You are constructing a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 10 meters, what dimensions will result in the window having the largest possible area (i.e. a maximum).

Note: Area of a circle = πr^2
Circumference of a circle = $2\pi r$



(2 marks each) Determine the 50th number in the sequence and the sum from the first number to the 50th number for each of the following series.

-25, -24, -23, -22, …

arithmetic series

$a_1 = -25$

$n = 50$

$K = (-24)-(-25) = 1$

$a_n = a_1 + (n-1)K$

$a_n = -25 + (50-1)(1)$

$\boxed{a_n = 24}$

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_n = \frac{50}{2}(-25 + 24)$

$\boxed{S_n = -25}$

1, 1.01, 1.0201, 1.030301, …

geometric series

$a_1 = 1$

$n = 50$

$r = \frac{1.01}{1} = 1.01$

$a_n = a_1 r^{n-1}$

$a_n = (1)(1.01^{50-1})$

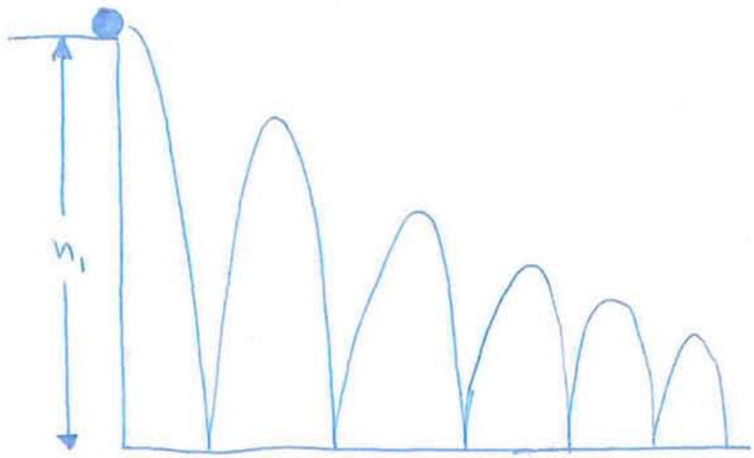
$\boxed{a_n = 1.63}$

$S_n = a_1 \frac{1 - r^n}{1 - r}$

$S_n = (1) \frac{1 - 1.01^{50}}{1 - 1.01}$

$\boxed{S_n = 64.5}$

(2 marks) If a ball falls from a table at h_1=1m and always rebounds to 97% of the previous bounce. How high will the ball bounce on the 50th rebound.



1, 0.97, 0.9409, 0.912673, ...

geometric series

$a_1 = 1$

$n = 50$

$r = \frac{0.97}{1} = 0.97$

$a_n = a_1 r^{n-1}$

$= (1)(0.97^{50-1})$

$= 0.225$

$\boxed{a_n = 0.23 \text{ m}}$

(2 marks) Take the derivative with respect to "x" of the following function **using the definition of the derivative.**

$$y(x) = ax + b$$

$$y(x+\Delta x) = a(x+\Delta x) + b$$

$$= ax + a\Delta x + b$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[ax + a\Delta x + b] - [ax + b]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{ax} + a\Delta x \cancel{+b} - \cancel{ax} - \cancel{b}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} a$$

$$= a$$

$$\boxed{\frac{dy}{dx} = a}$$

(2 marks each) Take the derivative with respect to “x” of the following functions.

$$y(x) = 2x^4 + sin(x)$$

$\frac{d}{dx}$

$$\boxed{\frac{dy}{dx} = 8x^3 + \cos(x)}$$

$$y(x) = 3x^2 sin(x)$$

$\frac{d}{dx}$

$$\boxed{\frac{dy}{dx} = 6x \sin(x) + 3x^2 \cos(x)}$$

$$y(x) = 3x^2 sin(3x^2+1) + x$$

$\frac{d}{dx}$

$$\boxed{\begin{aligned} \frac{dy}{dx} &= 6x\sin(3x^2+1) + 3x^2\cos(3x^2+1)(6x+0) + 1 \\ &= 6x\sin(3x^2+1) + 18x^3\cos(3x^2+1) + 1 \end{aligned}}$$

$$y(x) = \frac{3x^2}{sin(3x^2+1)} + x$$

$$y(x) = 3x^2\left(\sin(3x^2+1)\right)^{-1} + x$$

$\frac{d}{dx}$

$$\boxed{\begin{aligned} \frac{dy}{dx} &= 6x\left(\sin(3x^2+1)\right)^{-1} + 3x^2(-1)\left(\sin(3x^2+1)\right)^{-2}\left(\cos(3x^2+1)(6x+0)\right) \\ &\qquad + 1 \\ &= \frac{6x}{\sin(3x^2+1)} - \frac{18x^3\cos(3x^2+1)}{\left(\sin(3x^2+1)\right)^2} + 1 \\ &= \frac{6x\sin(3x^2+1) - 18x^3\cos(3x^2+1)}{\left(\sin(3x^2+1)\right)^2} + 1 \end{aligned}}$$

$$y(x) = (x-1)^2 sin((x-1)^3)$$

$\frac{d}{dx}$

$$\begin{aligned}
\frac{dy}{dx} &= 2(x-1)(1)\sin((x-1)^3) + (x-1)^2\cos((x-1)^3)(3(x-1)^2(1)) \\
&= 2(x-1)\sin((x-1)^3) + 3(x-1)^4\cos((x-1)^3)
\end{aligned}$$

$$y(x) = \left[\left((x-1)^3 - x\right)^3\right]^3$$

$\frac{d}{dx}$

$$\begin{aligned}
\frac{dy}{dx} &= 3\left[\left((x-1)^3-x\right)^3\right]^2\left[3\left((x-1)^3-x\right)^2\left(3(x-1)^2(1-0)-1\right)\right] \\
&= 9\left[\left((x-1)^3-x\right)^3\right]^2\left[\left((x-1)^3-x\right)^2\left(3(x-1)^2-1\right)\right] \\
&= 9\left[\left((x-1)^3-x\right)^3\right]^2\left((x-1)^3-x\right)^2\left(3(x-1)^2-1\right) \\
&= 9\left((x-1)^3-x\right)^6\left((x-1)^3-x\right)^2\left(3(x-1)^2-1\right) \\
&= 9\left((x-1)^3-x\right)^8\left(3(x-1)^2-1\right)
\end{aligned}$$

(5 marks) Determine the tangent line at x=-1 for the following function. Plot the tangent line on the plot.

$$y(x) = \sqrt{2^2 - x^2}$$

$$y(x) = \left(4 - x^2\right)^{\frac{1}{2}}$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = \frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$$\frac{dy(-1)}{dx} = \frac{-(-1)}{\sqrt{4-(-1)^2}} = 0.58 \leftarrow \text{slope @ } x=-1$$

$$y(-1) = \sqrt{4-(-1)^2} = 1.7 \leftarrow \text{y value @ } x=-1$$

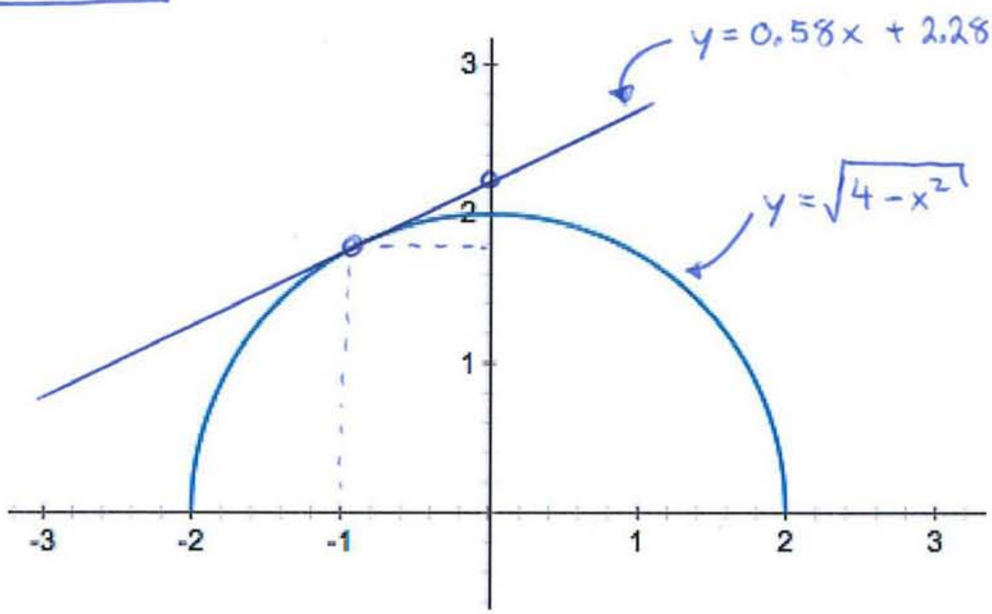
line equation

$$y = ax + b$$

$$b = y - ax = (1.7) - (0.58)(-1)$$

$$= 2.28$$

$$\boxed{y = 0.58x + 2.28}$$



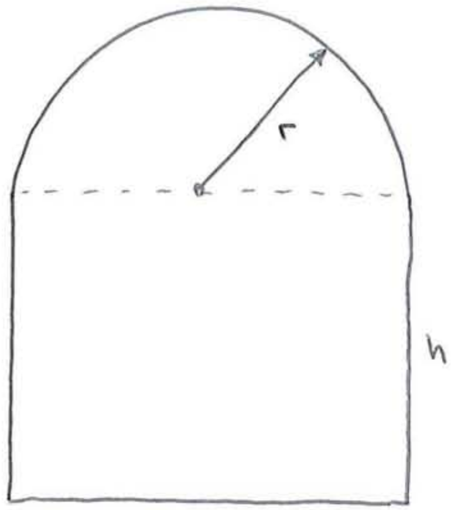
equation of a line:

$$y(x) = ax + b$$

(5 marks) You are constructing a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 10 meters, what dimensions will result in the window having the largest possible area (i.e. a maximum).

Note: Area of a circle = πr^2

Circumference of a circle = $2\pi r$



$$\text{Perimeter} = \pi r + 2r + 2h \qquad (1)$$

$$\text{Area} = 2rh + \tfrac{1}{2}\pi r^2 \qquad (2)$$

equation (1)

$$P = \pi r + 2r + 2h = r(\pi + 2) + 2h$$

$$\hookrightarrow h = \frac{P - 2h}{\pi + 2} \qquad (1^*)$$

sub (1*) into (2)

$$A = 2r\left(\frac{P - r(\pi+2)}{2}\right) + \frac{1}{2}\pi r^2$$

$$= \frac{\cancel{2}rP}{\cancel{2}} - \frac{\cancel{2}r^2(\pi+2)}{\cancel{2}} + \frac{1}{2}\pi r^2$$

$$= rP + r^2\left(\frac{1}{2}\pi - \pi - 2\right)$$

$$A = rP - r^2\left(\frac{\pi}{2} + 2\right)$$

$$\frac{d}{dx}\Big(\quad \frac{dA}{dx} = P\cancel{\frac{dr}{dr}} - 2r\cancel{\frac{dr}{dr}}\left(\frac{\pi}{2}\right) + 2\Big) = 0 \quad \text{max/min.}$$

$$-2r\left(\frac{\pi}{2} + 2\right) = -P$$

$$\boxed{r = \frac{P}{2\left(\frac{\pi}{2} + 2\right)} = \frac{P}{\pi + 4}} \qquad (3)$$

sub (3) into (1*)

$$h = \frac{P - \left(\frac{P}{\pi+4}\right)(\pi+2)}{2}$$

$$= \frac{P}{2} - \frac{P}{2}\frac{(\pi+2)}{(\pi+4)}$$

$$h = \frac{P}{2} - \frac{P}{2}\frac{(\pi+2)}{(\pi+4)}$$

$$= \frac{P(\pi+4)}{2(\pi+4)} - \frac{P(\pi+2)}{2(\pi+4)}$$

$$= \frac{\cancel{P\pi} + 4P - \cancel{P\pi} - 2P}{2(\pi+4)}$$

$$\boxed{h = \frac{2P}{2(\pi+4)} = \frac{P}{\pi+4} = r}$$

For $R = 10$

$$\boxed{\begin{array}{l} r = \frac{P}{\pi+4} = 1.40 \\ h = \frac{P}{\pi+4} = 1.40 \end{array}}$$