Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Test 1
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	5 questions worth 30 marks
Percentage of final grade:	23% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{x ln(a)}$$

Integrals of select functions

$$\int ax^n dx = \begin{cases} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{cases}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$ (exponentials)

$$\int ln(x)dx = xln(x) - x$$

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

(2 marks each) Determine the 100th number in the sequence and the sum from the first number to the 100th number for each of the following series.

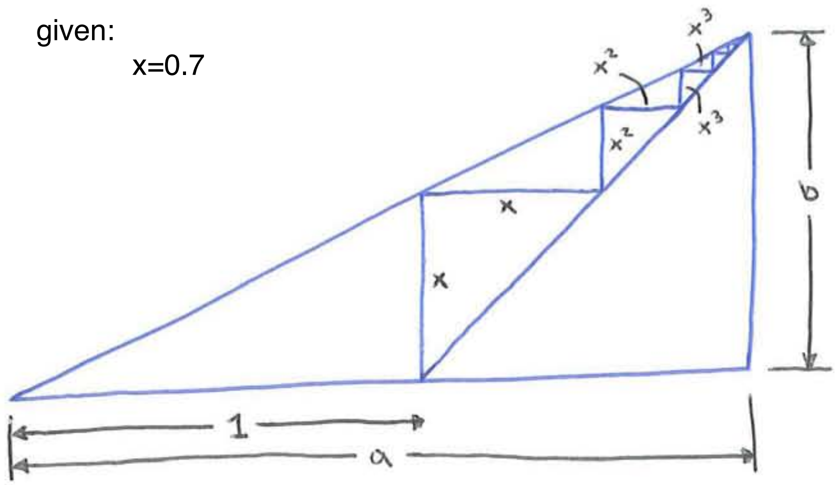
-5, -3.5, -2, -0.5, …

100, 99, 98.01, 97.0299, …

(4 marks) Use an infinite series to determine the lengths of “a” and “b” given the below figure. You may verify your answer using trigonometry, however, no marks will be given for a trigonometric solution. Set up an infinite series by adding the appropriate lengths to determine “a” and “b”.

given:

$x=0.7$



(2 marks each) Take the derivative with respect to "x" of the following functions.

$$y(x) = \frac{5}{2}x^2 + 2x^{\frac{5}{2}}$$

$$y(x) = 2x^5 \cos(x) + ax^k$$

$$y(x) = 3(x^2 - 1)^3 \sin(x^2) + \alpha$$

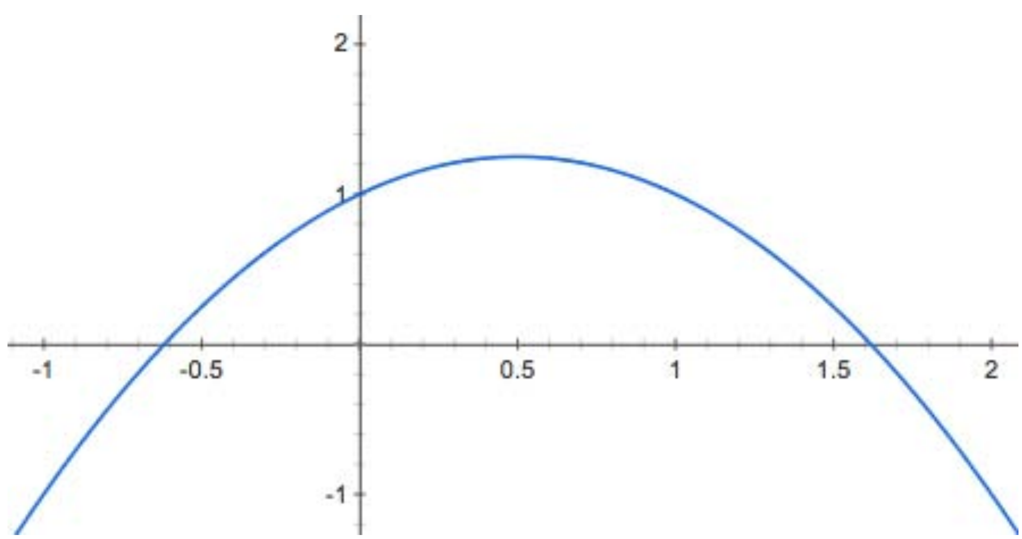
$$y(x) = \frac{3(x^2 - 1)^3}{2\sin(3x)} + \cos(\alpha x)$$

$$y^2 + sin(x) = x(y-1)^2 + \beta$$

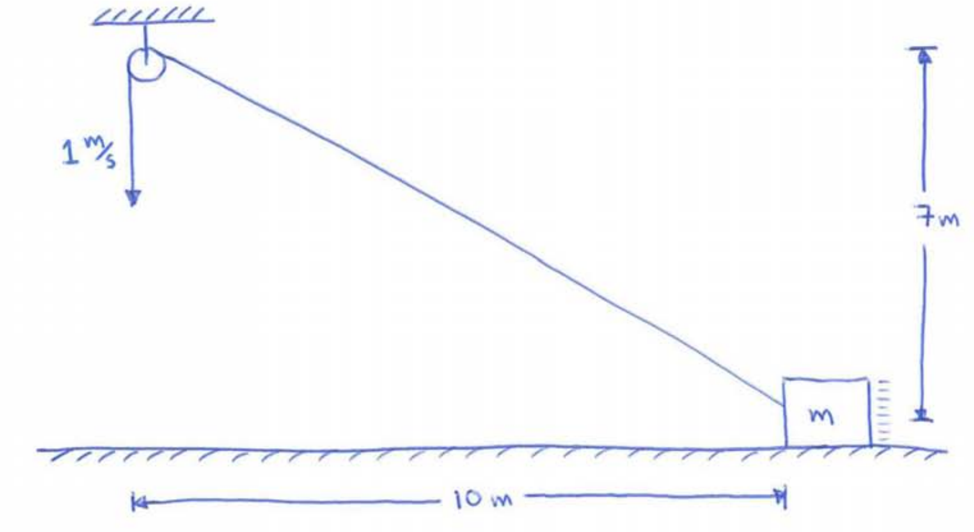
$$x^2 - 1 = 3\left[\sin\left(3(y^2-1)^2\right) + x\right]^3$$

(5 marks) Determine the tangent line at x=0 for the following function. Plot the tangent line on the plot.

$$y(x) = -x^2 + x + 1$$



(5 marks) Suppose you are dragging a mass (m) across the floor by the use of a rope and pulley, as shown in the below figure. If you are pulling the rope at a rate of 1 m/s how fast is the block sliding across the floor when at the position shown in the figure?



(2 marks each) Determine the 100th number in the sequence and the sum from the first number to the 100th number for each of the following series.

-5, -3.5, -2, -0.5, ...

$K = -0.5 - (-2) = 1.5$

$= -2 - (-3.5) = 1.5$ (arithmetric series)

$a_1 = -5$

$n = 100$

$a_n = a_1 + (n-1)k$

$a_{100} = -5 + (100-1)(1.5)$

$= 143.5$

$a_{100} = 143.5$

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_{100} = \frac{100}{2}(-5 + 143.5)$

$= 6925$

$S_{100} = 6925$

100, 99, 98.01, 97.0299, ...

$r = \frac{97.0299}{98.01} = 0.99$

$= \frac{98.01}{99} = 0.99$ (geometric series)

$a_1 = 100$

$n = 100$

$a_n = a_1 r^{n-1}$

$a_{100} = (100)(0.99^{100-1})$

$= 36.97$

$a_{100} = 36.97$

$S_n = a_1 \frac{1-r^n}{1-r}$

$S_{100} = (100) \frac{1-0.99^{100}}{1-0.99}$

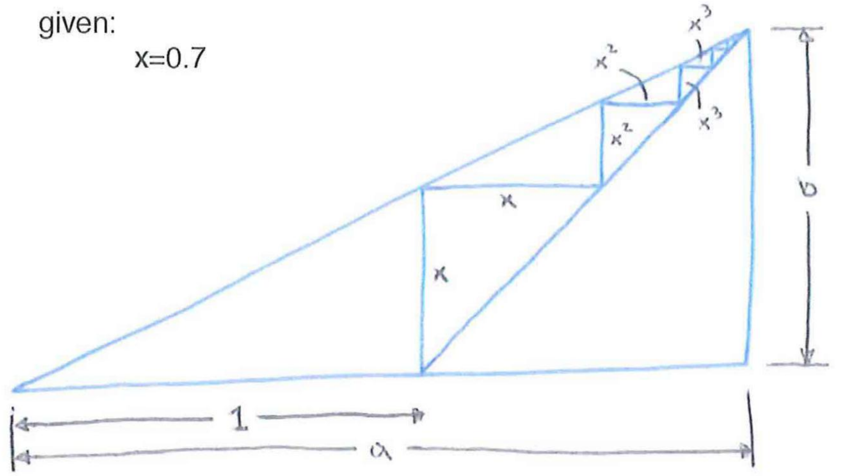
$= 6339.68$

$S_{100} = 6339.68$

(4 marks) Use an infinite series to determine the lengths of “a” and “b” given the below figure. You may verify your answer using trigonometry, however, no marks will be given for a trigonometric solution. Set up an infinite series by adding the appropriate lengths to determine “a” and “b”.

given:

x=0.7



From Figure:

$$a = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$b = x + x^2 + x^3 + x^4 + \dots$$

Geometric series for a:

$1, x, x^2, x^3, \dots$

$r = \frac{x^3}{x^2} = \frac{x^2}{x} = x$ (geometric)

$a_1 = 1$

$n \to \infty$

so

$$S_n = a_1 \frac{1-r^n}{1-r}$$

$$= (1)\frac{1-x^{\infty}}{1-x} \quad (x^{\infty} \to 0 \text{ for } -1<x<1)$$

$$= \frac{1}{1-x}$$

For $x = 0.7$

$$S_n = \frac{1}{1-0.7} = 3.33$$

$a = 3.33$

Geometric series for b:

$x, x^2, x^3, x^4, \dots$

$r = \frac{x^3}{x^2} = \frac{x^2}{x} = x$ (geometric)

$a_1 = x$

$n \to \infty$

so

$$S_n = a_1 \frac{1-r^n}{1-r}$$

$$= (x)\frac{1-x^{\infty}}{1-x} \quad (x^{\infty} \to 0 \text{ for } -1<x<1)$$

$$= \frac{x}{1-x}$$

For $x = 0.7$

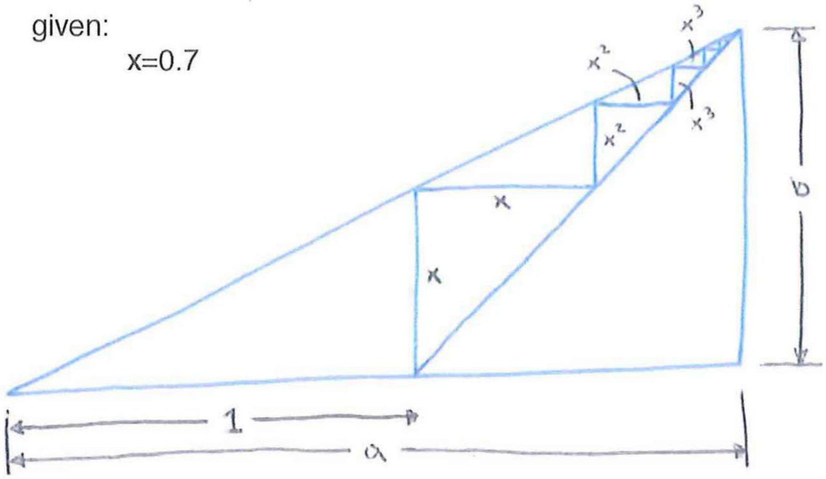
$$S_n = \frac{0.7}{1-0.7} = 2.33$$

$b = 2.33$

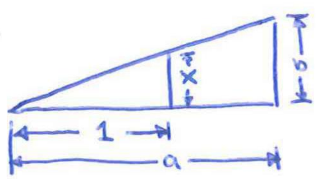
(4 marks) Use an infinite series to determine the lengths of “a” and “b” given the below figure. You may verify your answer using trigonometry, however, no marks will be given for a trigonometric solution. Set up an infinite series by adding the appropriate lengths to determine “a” and “b”.

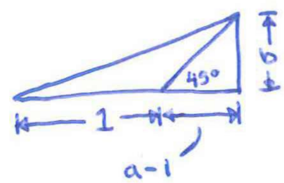
given:

x=0.7



Using Trigonometry:





Similar triangles

$$\boxed{\frac{a}{1} = \frac{b}{x}} \qquad (1)$$

Inner triangle @ 45°

$$\boxed{a - 1 = b} \qquad (2)$$

sub (1) into (2)

$$\left[\frac{b}{x}\right] - 1 = b$$

$$b - x = bx$$

$$b - bx = x$$

$$b(1-x) = x$$

$$b = \frac{x}{1-x}$$

sub back into (1)

$$a = \frac{\left[x/1-x\right]}{x} = \frac{1}{1-x}$$

for $x = 0.7$

$$a = \frac{1}{1-0.7} = 3.33$$

$$b = \frac{0.7}{1-0.7} = 2.33$$

(2 marks each) Take the derivative with respect to "x" of the following functions.

$$y(x) = \frac{5}{2}x^2 + 2x^{\frac{5}{2}}$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = 5x + 5x^{\frac{3}{2}}$$

$$y(x) = 2x^5 \cos(x) + ax^k$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = 10x^4 \cos(x) - 2x^5 \sin(x) + akx^{k-1}$$

$$y(x) = 3(x^2 - 1)^3 \sin(x^2) + \alpha$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = 9(x^2-1)^2(2x)\sin(x^2) + 3(x^2-1)^3\cos(x^2)(2x)$$

$$y(x) = \frac{3(x^2-1)^3}{2\sin(3x)} + \cos(\alpha x)$$

$$y(x) = \left[3(x^2-1)^3\right]\left[2\sin(3x)\right]^{-1} + \cos(\alpha x)$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = \left[9(x^2-1)^2(2x)\right]\left[2\sin(3x)\right]^{-1} + \left[3(x^2-1)^3\right]\left[(-1)(2\sin(3x))^{-2}(6\cos(3x))\right] - \alpha\sin(\alpha x)$$

$$= \frac{\left[9(x^2-1)^2(2x)\right]\left[2\sin(3x)\right] - \left[3(x^2-1)^3\right]\left[6\cos(3x)\right]}{\left[2\sin(3x)\right]^2} - \alpha\sin(\alpha x)$$

$$y^2 + sin(x) = x(y-1)^2 + \beta$$

$\frac{d}{dx}$

$$2y\left(\frac{dy}{dx}\right) + \cos(x) = (1)(y-1)^2 + (x)\left(2(y-1)\left(\frac{dy}{dx}\right)\right)$$

$$x^2 - 1 = 3\left[\sin\left(3(y^2-1)^2\right) + x\right]^3$$

$\frac{d}{dx}$

$$2x = 9\left[\sin\left(3(y^2-1)^2\right) + x\right]^2\left[\cos\left(3(y^2-1)^2\right)\left(6(y^2-1)\left(2y\left(\frac{dy}{dx}\right)\right)\right) + 1\right]$$

(5 marks) Determine the tangent line at x=0 for the following function. Plot the tangent line on the plot.

$$y(x) = -x^2 + x + 1$$

y value @ x=0

$y(0) = -(0)^2 + (0) + 1 = 1$

derivative of Function

$y(x) = -x^2 + x + 1$

$\frac{d}{dx}$

$\frac{dy}{dx} = -2x + 1$

slope @ x=0

$\frac{dy}{dx} = -2(0) + 1 = 1$

equation of a line:

$y = ax + b$

sub a=1

$y = x + b$

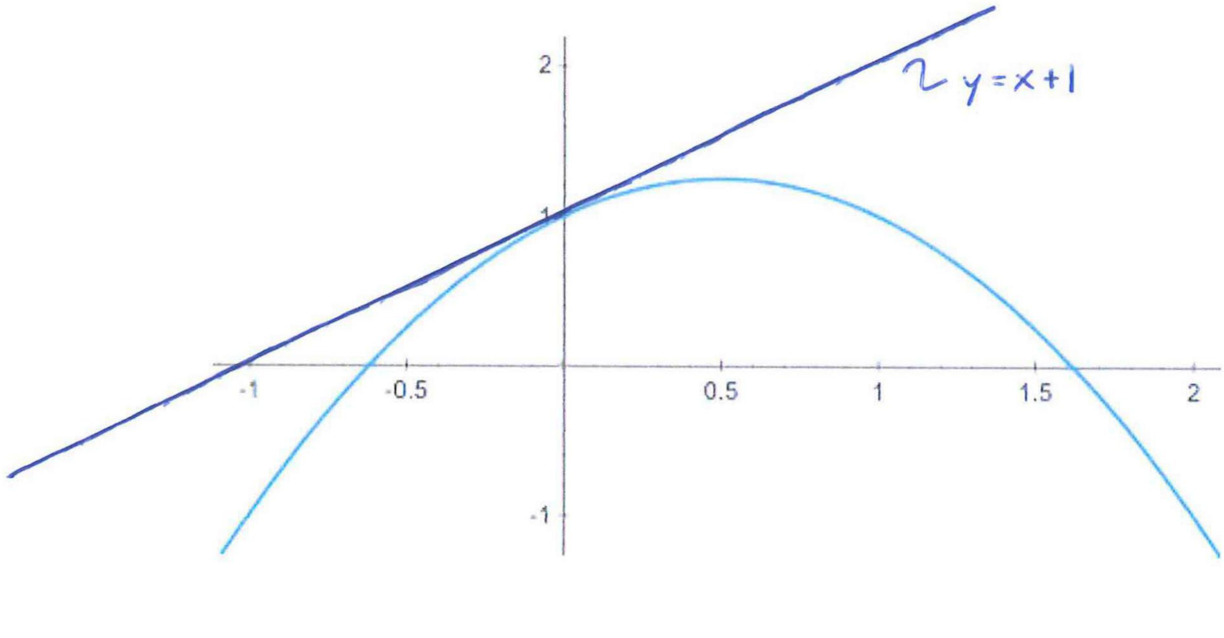
sub y=1 @ x=0

$1 = 0 + b$

$b = 1$

so

$\boxed{y = x + 1}$



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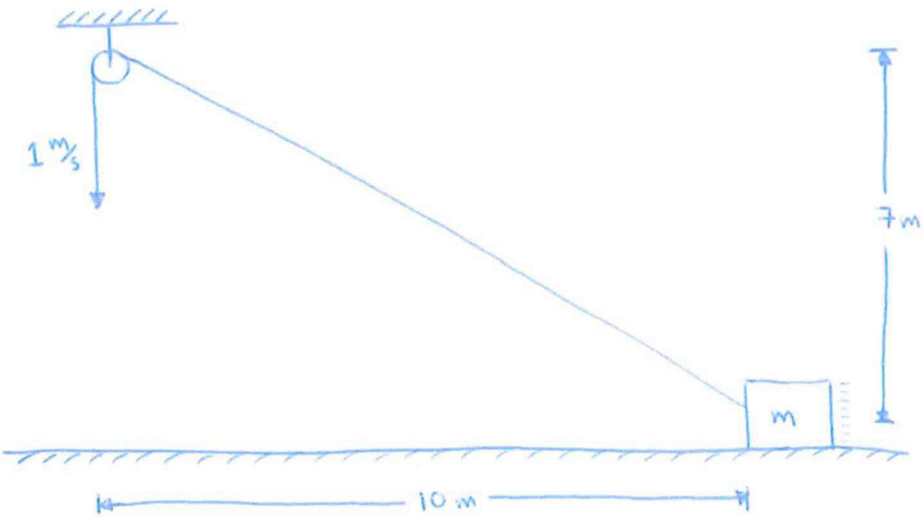

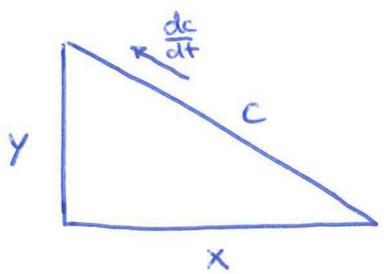

Pythagorean theorem:

$$x^2 + y^2 = c^2$$

@ $x = 10,\ y = 7$

$$c = \sqrt{x^2 + y^2}$$

$$= \sqrt{10^2 + 7^2}$$

$$= 12.2 \text{ m}$$

take derivative with respect to time

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2c \frac{dc}{dt}$$

($\frac{dy}{dt} = 0$ (not moving up/down))

$$\frac{dx}{dt} = \frac{c}{x} \frac{dc}{dt}$$

$$= \frac{12.2}{10}(1)$$

$$= 1.22$$

$$\frac{dx}{dt} = 1.22 \text{ m/s}$$