

Instructor:	Frank Secretain
Course:	Math 20
Assessment:	Test 1
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	6 questions worth 30 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n - 1)k$$

$$S_n = \sum_{i=1}^n a_1 + (i - 1)k$$
$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^n a_1 r^{i-1}$$
$$= a_1 \frac{1 - r^n}{1 - r}$$

Binomial Theorem

$$(x + y)^n =$$

$$\sum_{k=0}^n \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x) \frac{d}{dx}(g(x)) + g(x) \frac{d}{dx}(f(x)) \quad \text{(product rule)}$$

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{d}{dx}(f(x)) - f(x) \frac{d}{dx}(g(x))}{(g(x))^2} \quad \text{(quotient rule)}$$

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x))) \frac{d}{dx}(g(x)) \quad \text{(chain rule)}$$

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \frac{1}{(\cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln(a)}$$

Integrals of select functions

$$\int ax^n dx = \left\{ \begin{array}{ll} \frac{a}{n+1} x^{n+1} & , n \neq -1 \\ \ln(|x|) & , n = -1 \end{array} \right\} \quad \text{(polynomials)}$$

$$\int \sin(ax) dx = -\frac{1}{a} \cos(ax)$$

$$\int \cos(ax) dx = \frac{1}{a} \sin(ax) \quad \text{(trigonometry)}$$

$$\int \tan(ax) dx = \ln(|\sec(x)|)$$

$$\int a^x dx = \frac{1}{\ln(a)} a^x$$

$$\int \ln(x) dx = x \ln(x) - x \quad \text{(exponentials)}$$

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!} (x - x_o)^n$$

Integration by parts

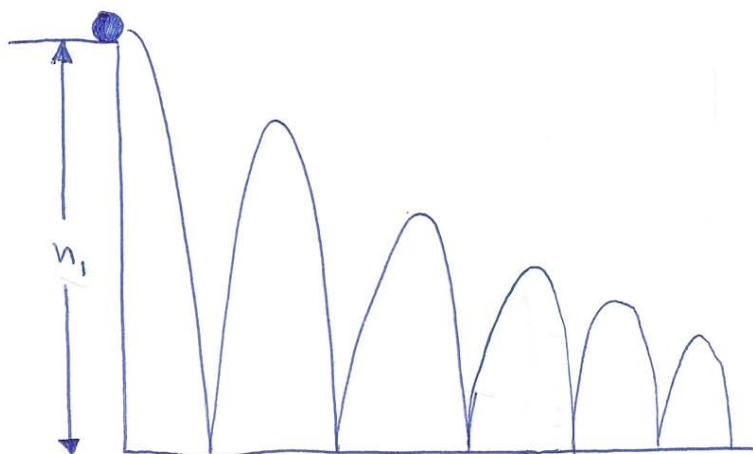
$$\int u dv = uv - \int v du$$

(2 marks each) Determine the 50th number in the sequence and the sum from the first number to the 50th number for each of the following series.

-25, -24, -23, -22, ...

1, 1.01, 1.0201, 1.030301, ...

(2 marks) If a ball falls from a table at $h_1=1\text{m}$ and always rebounds to 97% of the previous bounce. How high will the ball bounce on the 50th rebound.



(2 marks) Take the derivative with respect to “x” of the following function **using the definition of the derivative**.

$$y(x) = ax + b$$

(2 marks each) Take the derivative with respect to “x” of the following functions.

$$y(x) = 2x^4 + \sin(x)$$

$$y(x) = 3x^2 \sin(x)$$

$$y(x) = 3x^2 \sin(3x^2 + 1) + x$$

$$y(x) = \frac{3x^2}{\sin(3x^2 + 1)} + x$$

$$y(x) = (x - 1)^2 \sin((x - 1)^3)$$

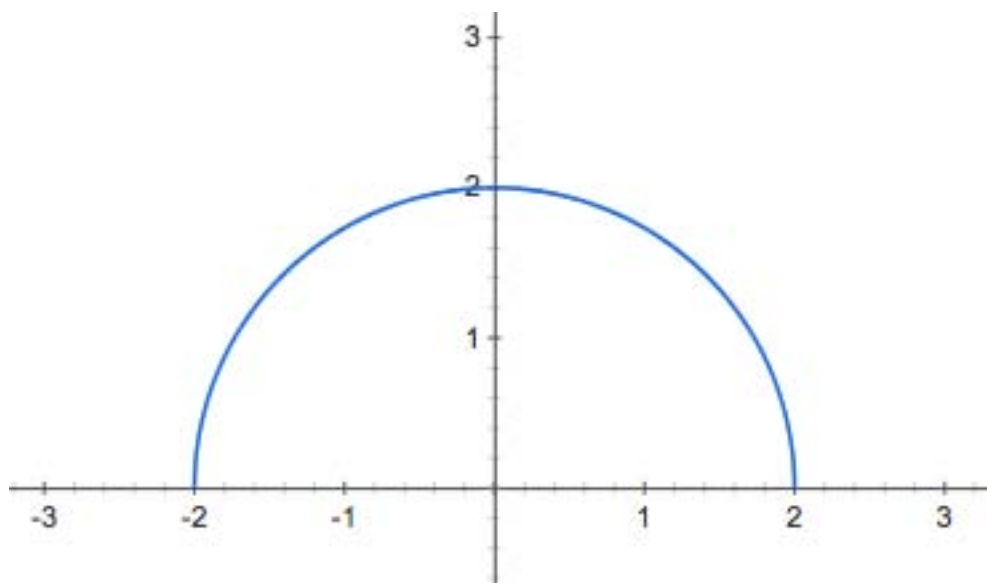
$$y(x) = \left[((x - 1)^3 - x)^3 \right]^3$$

(5 marks) Determine the tangent line at $x=-1$ for the following function. Plot the tangent line on the plot.

$$y(x) = \sqrt{2^2 - x^2}$$

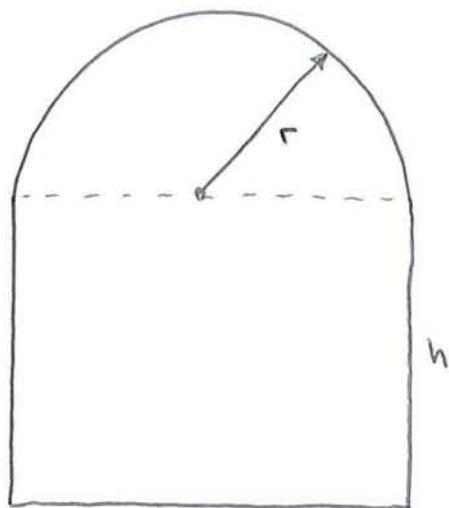
equation of a line:

$$y(x) = ax + b$$



(5 marks) You are constructing a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 10 meters, what dimensions will result in the window having the largest possible area (i.e. a maximum).

Note: Area of a circle = πr^2
Circumference of a circle = $2\pi r$



(2 marks each) Determine the 50th number in the sequence and the sum from the first number to the 50th number for each of the following series.

-25, -24, -23, -22, ...

arithmetic series

$$a_1 = -25$$

$$n = 50$$

$$k = (-24) - (-25) = 1$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{50}{2}(-25 + 24)$$

$$S_n = -25$$

$$a_n = a_1 + (n-1)k$$

$$a_n = -25 + (50-1)(1)$$

$$a_n = 24$$

1, 1.01, 1.0201, 1.030301, ...

geometric series

$$a_1 = 1$$

$$n = 50$$

$$r = \frac{1.01}{1} = 1.01$$

$$S_n = a_1 \frac{1-r^n}{1-r}$$

$$S_n = (1) \frac{1-1.01^{50}}{1-1.01}$$

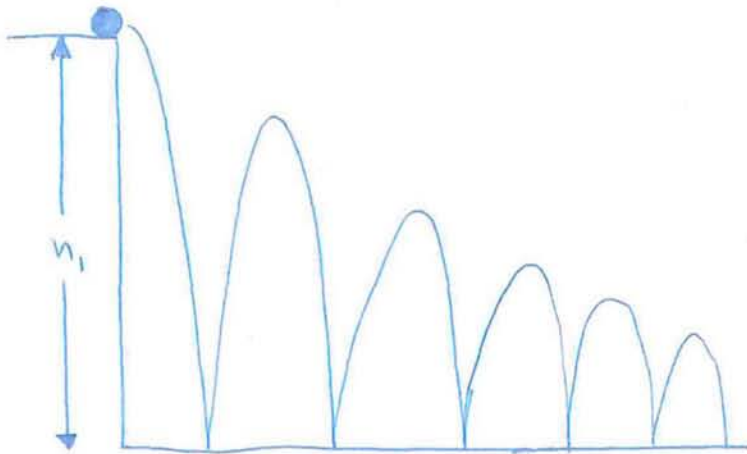
$$S_n = 64.5$$

$$a_n = a_1 r^{n-1}$$

$$a_n = (1)(1.01^{50-1})$$

$$a_n = 1.63$$

(2 marks) If a ball falls from a table at $h_1=1\text{ m}$ and always rebounds to 97% of the previous bounce. How high will the ball bounce on the 50th rebound.



1, 0.97, 0.9409, 0.912673, ...

geometric series

$$a_1 = 1$$

$$n = 50$$

$$r = \frac{0.97}{1} = 0.97$$

$$a_n = a_1 r^{n-1}$$

$$= (1)(0.97^{50-1})$$

$$= 0.225$$

$$a_n = 0.23 \text{ m}$$

(2 marks) Take the derivative with respect to "x" of the following function using the definition of the derivative.

$$y(x) = ax + b$$

$$\begin{aligned} y(x+\Delta x) &= a(x+\Delta x) + b \\ &= ax + a\Delta x + b \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[ax + a\Delta x + b] - [ax + b]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cancel{ax} + a\Delta x + \cancel{b} - \cancel{ax} - \cancel{b}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{a\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} a \\ &= a \end{aligned}$$

$$\boxed{\frac{dy}{dx} = a}$$

(2 marks each) Take the derivative with respect to "x" of the following functions.

$$y(x) = 2x^4 + \sin(x)$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = 8x^3 + \cos(x)$$

$$y(x) = 3x^2 \sin(x)$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = 6x \sin(x) + 3x^2 \cos(x)$$

$$y(x) = 3x^2 \sin(3x^2 + 1) + x$$

$\frac{d}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 6x \sin(3x^2 + 1) + 3x^2 \cos(3x^2 + 1)(6x + 0) + 1 \\ &= 6x \sin(3x^2 + 1) + 18x^3 \cos(3x^2 + 1) + 1 \end{aligned}$$

$$y(x) = \frac{3x^2}{\sin(3x^2 + 1)} + x$$

$$y(x) = 3x^2 (\sin(3x^2 + 1))^{-1} + x$$

$\frac{d}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 6x (\sin(3x^2 + 1))^{-1} + 3x^2 (-1) (\sin(3x^2 + 1))^{-2} (\cos(3x^2 + 1)(6x + 0)) \\ &\quad + 1 \\ &= \frac{6x}{\sin(3x^2 + 1)} - \frac{18x^3 \cos(3x^2 + 1)}{(\sin(3x^2 + 1))^2} + 1 \\ &= \frac{6x \sin(3x^2 + 1) - 18x^3 \cos(3x^2 + 1)}{(\sin(3x^2 + 1))^2} + 1 \end{aligned}$$

$$y(x) = (x-1)^2 \sin((x-1)^3)$$

$\frac{d}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 2(x-1)(1) \sin((x-1)^3) + (x-1)^2 \cos((x-1)^3)(3(x-1)^2(1)) \\ &= 2(x-1) \sin((x-1)^3) + 3(x-1)^4 \cos((x-1)^3) \end{aligned}$$

$$y(x) = \left[((x-1)^3 - x)^3 \right]^3$$

$\frac{d}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 3 \left[((x-1)^3 - x)^3 \right]^2 \left[3((x-1)^3 - x)^2 (3(x-1)^2(1-0) - 1) \right] \\ &= 9 \left[((x-1)^3 - x)^3 \right]^2 \left[((x-1)^3 - x)^2 (3(x-1)^2 - 1) \right] \\ &= 9 \left[((x-1)^3 - x)^3 \right]^2 ((x-1)^3 - x)^2 (3(x-1)^2 - 1) \\ &= 9 ((x-1)^3 - x)^6 ((x-1)^3 - x)^2 (3(x-1)^2 - 1) \\ &= 9 ((x-1)^3 - x)^8 (3(x-1)^2 - 1) \end{aligned}$$

(5 marks) Determine the tangent line at $x=-1$ for the following function. Plot the tangent line on the plot.

$$y(x) = \sqrt{2^2 - x^2}$$

$$y(x) = (4 - x^2)^{\frac{1}{2}}$$

$\frac{d}{dx}$

$$\frac{dy}{dx} = \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{4 - x^2}}$$

$$\frac{dy(-1)}{dx} = \frac{-(-1)}{\sqrt{4 - (-1)^2}} = 0.58 \quad \leftarrow \text{slope @ } x = -1$$

$$y(-1) = \sqrt{4 - (-1)^2} = 1.7 \quad \leftarrow y \text{ value @ } x = -1$$

$$y = 0.58x + 2.28$$

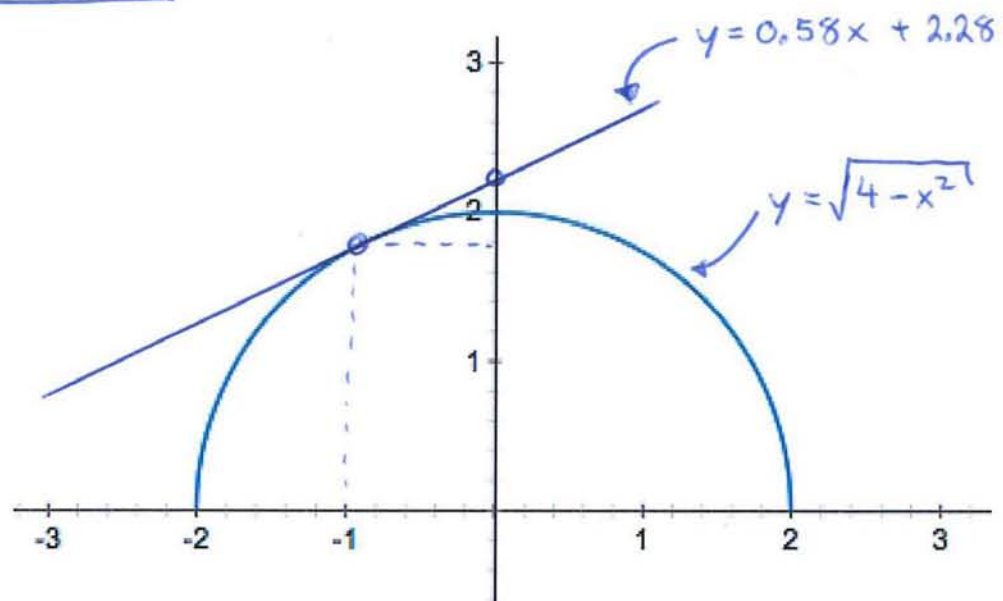
equation of a line:

$$y(x) = ax + b$$

line equation

$$y = ax + b$$

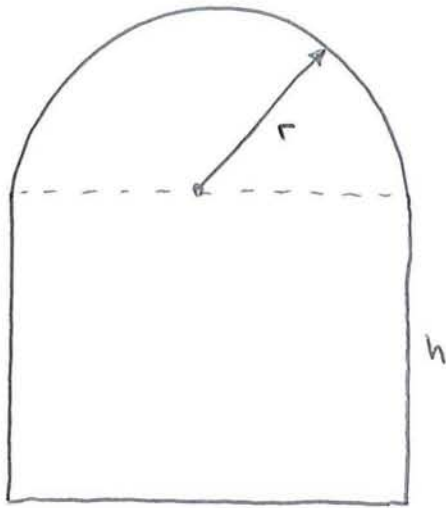
$$b = y - ax = (1.7) - (0.58)(-1) = 2.28$$



(5 marks) You are constructing a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 10 meters, what dimensions will result in the window having the largest possible area (i.e. a maximum).

Note: Area of a circle = πr^2

Circumference of a circle = $2\pi r$



$$\text{Perimeter} = \pi r + 2r + 2h \quad (1)$$

$$\text{Area} = 2rh + \frac{1}{2}\pi r^2 \quad (2)$$

equation (1)

$$P = \pi r + 2r + 2h = r(\pi + 2) + 2h$$

$$\rightarrow h = \frac{P - 2h}{\pi + 2} \quad (1^*)$$

sub (1*) into (2)

$$A = 2r \left(\frac{P - r(\pi + 2)}{2} \right) + \frac{1}{2}\pi r^2$$

$$= \frac{2rP}{2} - \frac{2r^2(\pi + 2)}{2} + \frac{1}{2}\pi r^2$$

$$= rP + r^2 \left(\frac{1}{2}\pi - \pi - 2 \right)$$

$$A = rP - r^2 \left(\frac{\pi}{2} + 2 \right)$$

$$\frac{d}{dx} \left(\frac{dA}{dx} = P - 2r \left(\frac{\pi}{2} + 2 \right) \right) = 0 \quad \text{max/min.}$$

$$-2r \left(\frac{\pi}{2} + 2 \right) = -P$$

$$r = \frac{P}{2 \left(\frac{\pi}{2} + 2 \right)} = \frac{P}{\pi + 4} \quad (3)$$

sub (3) into (1*)

$$h = \frac{P - \left(\frac{P}{\pi + 4} \right)(\pi + 2)}{2}$$

$$= \frac{P}{2} - \frac{P}{2} \frac{(\pi + 2)}{(\pi + 4)}$$

$$\begin{aligned} h &= \frac{P}{2} - \frac{P}{2} \frac{(\pi + 2)}{(\pi + 4)} \\ &= \frac{P(\pi + 4)}{2(\pi + 4)} - \frac{P(\pi + 2)}{2(\pi + 4)} \\ &= \frac{P\pi + 4P - P\pi - 2P}{2(\pi + 4)} \end{aligned}$$

$$h = \frac{2P}{2(\pi + 4)} = \frac{P}{\pi + 4} = r$$

For $P = 10$

$$r = \frac{P}{\pi + 4} = 1.40$$

$$h = \frac{P}{\pi + 4} = 1.40$$