Instructor:	Frank Secretain
Course:	Math 20
Date:	November 28, 2024
Assessment:	Test 3
Time allowed:	110 minutes
Devices allowed:	Pencil, pen, eraser, calculator
Notes from instructor:	Be neat. Show your work where needed. Box final answers.
Marks allocated:	3 questions worth 20 marks
Percentage of final grade:	20% of final grade

Formula Sheet

Arithmetic Series

$$a_n = a_1 + (n-1)k$$

$$S_n = \sum_{i=1}^{n} a_1 + (i-1)k$$

$$= \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$a_n = a_1 r^{n-1}$$

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1}$$

$$= a_1 \frac{1-r^n}{1-r}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$$

Line equation

$$y = ax + b$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of the derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Rules of differentiation

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$ (product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$ (quotient rule)

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))$$ (chain rule)

Derivatives of select functions

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = \frac{1}{(cos(x))^2}$$

$$\frac{d}{dx}(a^x) = a^x ln(a)$$

$$\frac{d}{dx}(log_a x) = \frac{1}{xln(a)}$$

Integrals of select functions

$$\int ax^n dx = \left\{ \begin{array}{ll} \frac{a}{n+1}x^{n+1} & , n \neq -1 \\ ln(|x|) & , n = -1 \end{array} \right\}$$ (polynomials)

$$\int sin(ax)dx = -\frac{1}{a}cos(ax)$$

$$\int cos(ax)dx = \frac{1}{a}sin(ax)$$ (trigonometry)

$$\int tan(ax)dx = ln(|sec(x)|)$$

$$\int a^x dx = \frac{1}{ln(a)}a^x$$

$$\int ln(x)dx = xln(x) - x$$ (exponentials)

Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n$$

Integration by parts

$$\int udv = uv - \int vdu$$

(2 marks each) Integrate the following:

$$\int 4x^3 - x \ dx$$

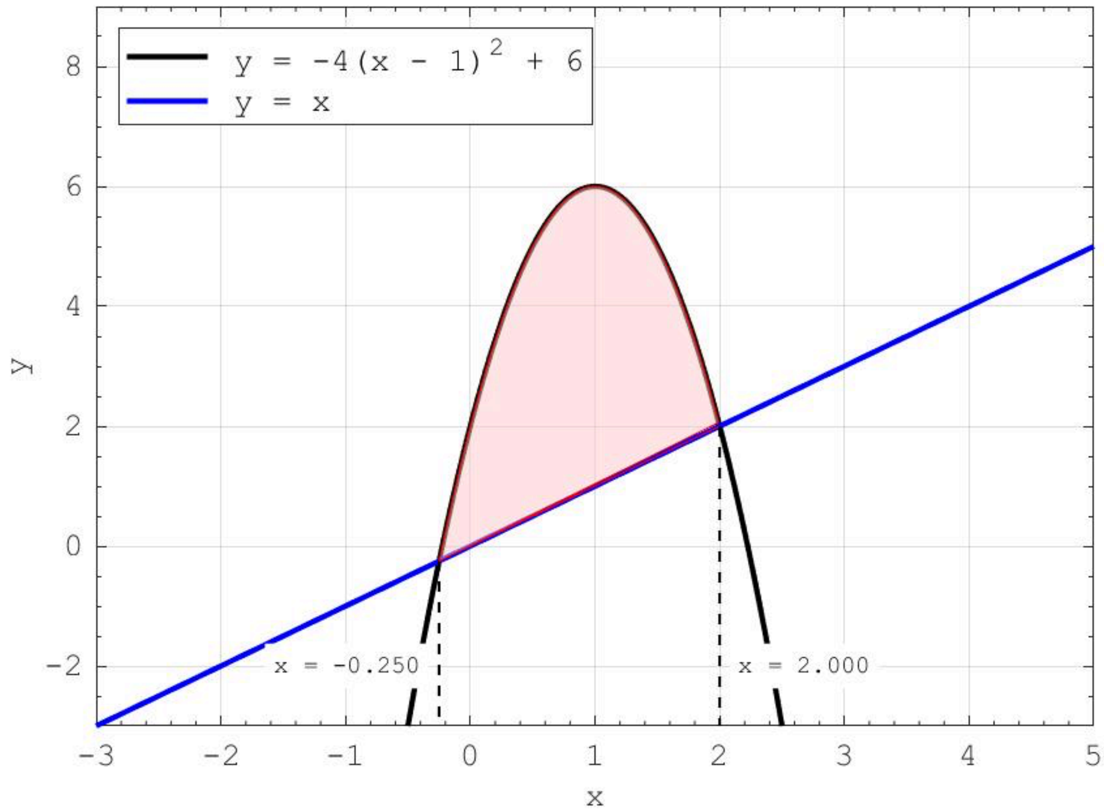
$$\int 3\sin(2x - \phi) + b \ dx$$

$$\int 3x^2 \cos(x^3 + b)\ dx$$

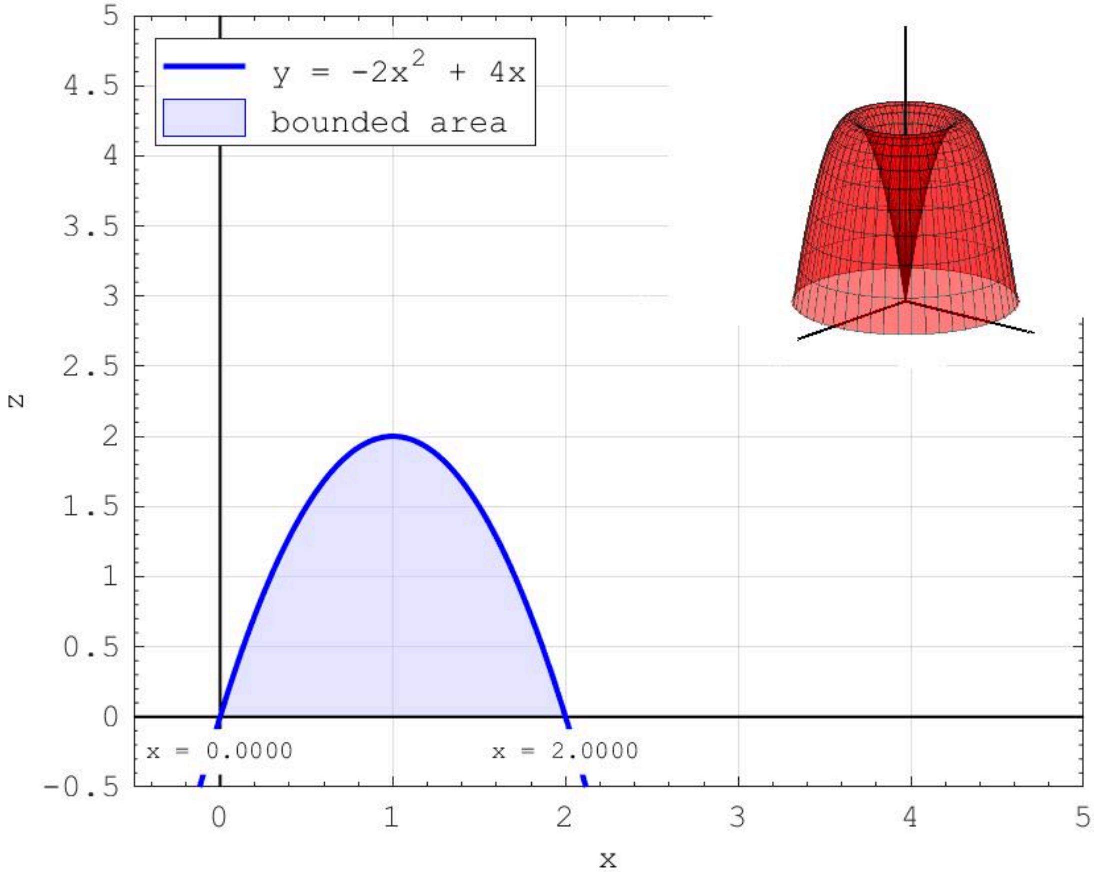
$$\int \frac{3\alpha}{x^2} - x^2\ dx$$

$$\int 2x \sin(2x)\ dx$$

(5 marks) Determine the shaded area of the two functions given in the legend.



(5 marks) Calculate the volume of the bounded area of the function given in the legend (shaded in blue) revolved around the z-axis (as shown in the top left corner shaded in red).



(2 marks each) Integrate the following:

$$\int 4x^3 - x \, dx$$

$$= \boxed{x^4 - \frac{1}{2}x^2 + c}$$

$$\int 3\sin(2x - \phi) + b \, dx$$

$$= \boxed{-\frac{3}{2}\cos(2x - \phi) + bx + c}$$

$$\int 3x^2 \cos(x^3 + b)\ dx$$

let $u = x^3 + b$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$$\int \cancel{3x^2} \cos(u) \frac{du}{\cancel{3x^2}}$$

$= \sin(u) + c$

$= \boxed{\sin(x^3 + b) + c}$

$$\int \frac{3\alpha}{x^2} - x^2\ dx$$

$$\int 3\alpha x^{-2} - x^2\ dx$$

$= -3\alpha x^{-1} - \frac{1}{3} x^3 + c$

$= \boxed{-\frac{3\alpha}{x} - \frac{x^3}{3} + c}$

$$\int 2x\sin(2x)\ dx$$

let $u = 2x$ $\quad dv = \sin(2x)\,dx$

$\frac{du}{dx} = 2$ $\quad v = -\frac{1}{2}\cos(2x)$

$du = 2\,dx$

$$\int 2x\sin(2x)\,dx = \int u\,dv = uv - \int v\,du$$

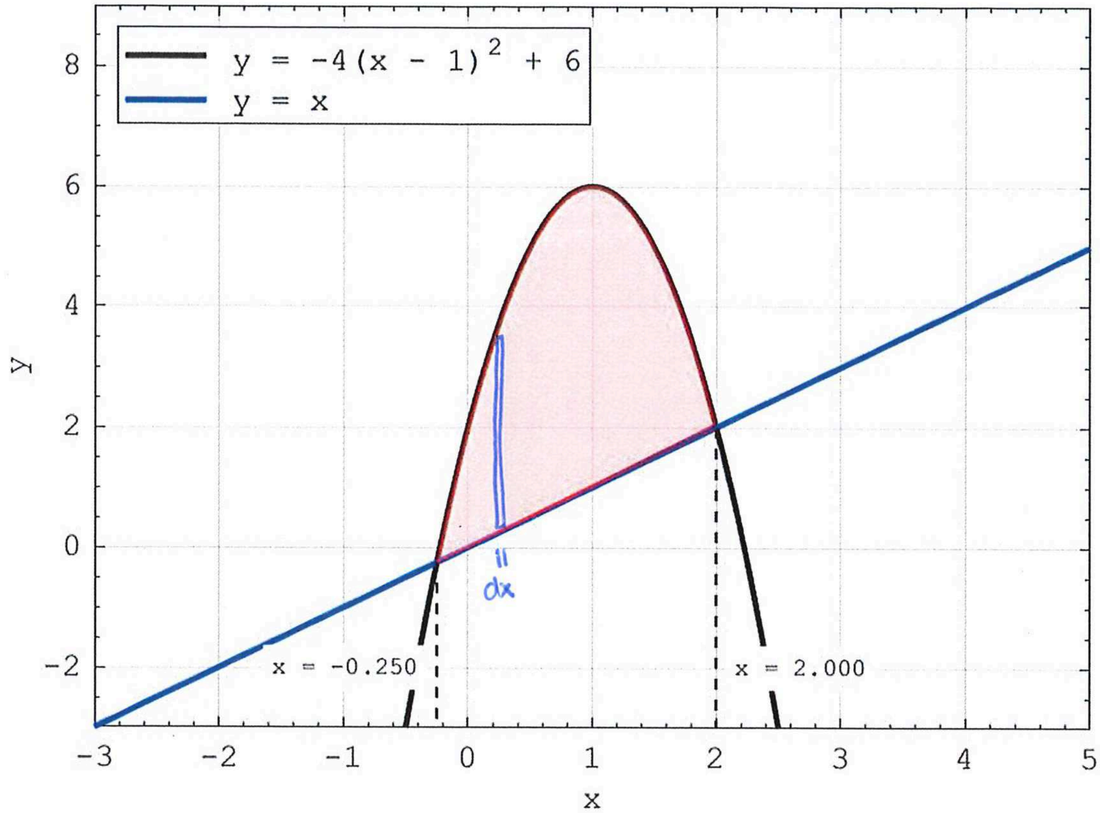
$$= [2x]\left[-\tfrac{1}{2}\cos(2x)\right] - \int -\tfrac{1}{2}\cos(2x)\ 2\,dx$$

$$= -x\cos(2x) + \int \cos(2x)\,dx$$

$$= -x\cos(2x) + \left[\tfrac{1}{2}\sin(2x)\right] + c$$

$$= \boxed{-x\cos(2x) + \frac{1}{2}\sin(2x) + c}$$

(5 marks) Determine the shaded area of the two functions given in the legend.



columns:

$$dA = h\,dx = \left(\left[-4(x-1)^2 + 6\right] - \left[x\right]\right) dx$$

$$\int dA = \int_{-0.25}^{2} -4(x-1)^2 + 6 - x \; dx$$

$$A = \left[-\frac{4}{3}(x-1)^3 + 6x - \frac{1}{2}x^2\right]_{-0.25}^{2}$$

$$= \left[-\frac{4}{3}([2]-1)^3 + 6[2] - \frac{1}{2}[2]^2\right] - \left[-\frac{4}{3}([-0.25]-1)^3 + 6[-0.25] - \frac{1}{2}[-0.25]^2\right]$$

$$= \left[-\frac{4}{3} + 12 - 2\right] - \left[2.6042 - 1.5 - 0.03125\right]$$

$$\boxed{A = 7.59}$$

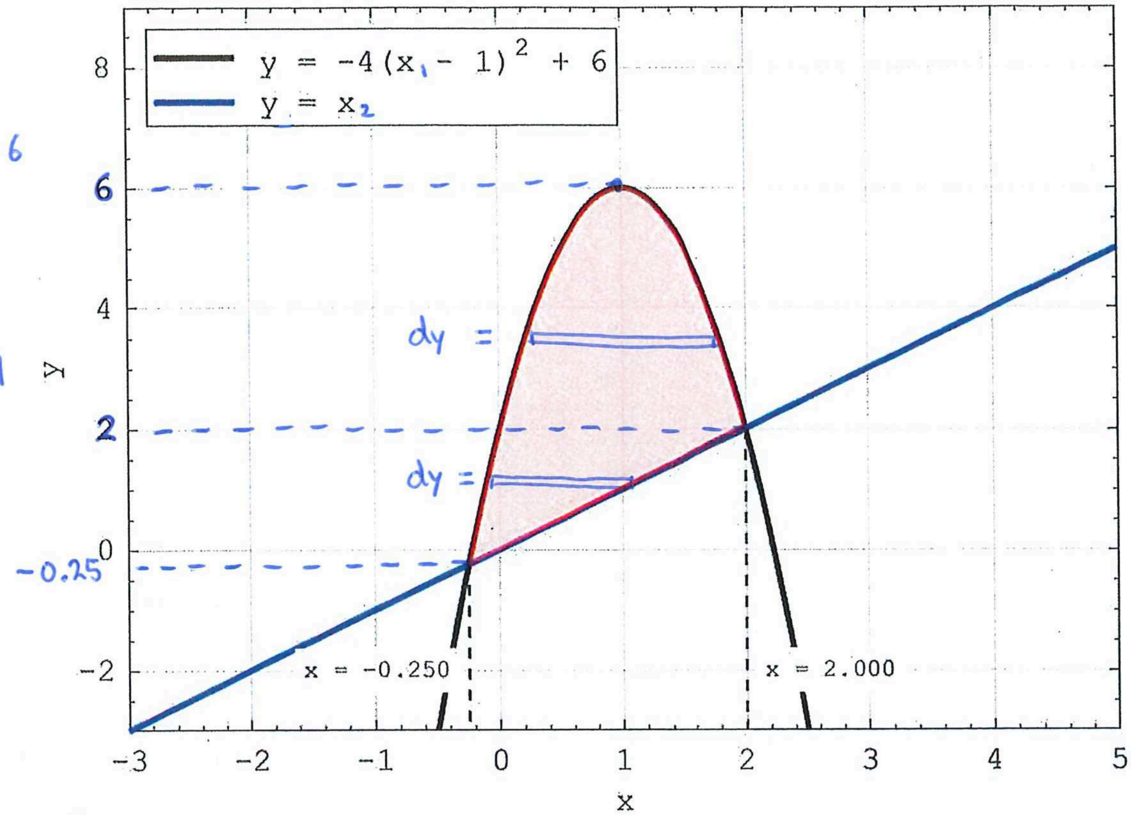
(5 marks) Determine the shaded area of the two functions given in the legend.

solve for x_i:

$y = -4(x-1)^2 + 6$

$(x-1)^2 = \frac{6-y}{4}$

$x = \pm\sqrt{\frac{6-y}{4}} + 1$



rows:

$$dA = w\,dy = \int (x_2 - x_1)\,dy + \int (x_{1+} - x_{1-})\,dy$$

$$= \int_{-0.25}^{2} \left([y] - \left[-\sqrt{\frac{6-y}{4}} + 1\right]\right) dy + \int_{2}^{6} \left(\left[+\sqrt{\frac{6-y}{4}} + 1\right] - \left[-\sqrt{\frac{6-y}{4}} + 1\right]\right) dy$$

$$= \left[\frac{y^2}{2} - \frac{8}{3}\left(\frac{6-y}{4}\right)^{\frac{3}{2}} - y\right]_{-0.25}^{2} + \left[-\frac{8}{3}\left(\frac{6-y}{4}\right)^{\frac{3}{2}} + \cancel{y} - \frac{8}{3}\left(\frac{6-y}{4}\right)^{\frac{3}{2}} - \cancel{y}\right]_{2}^{6}$$

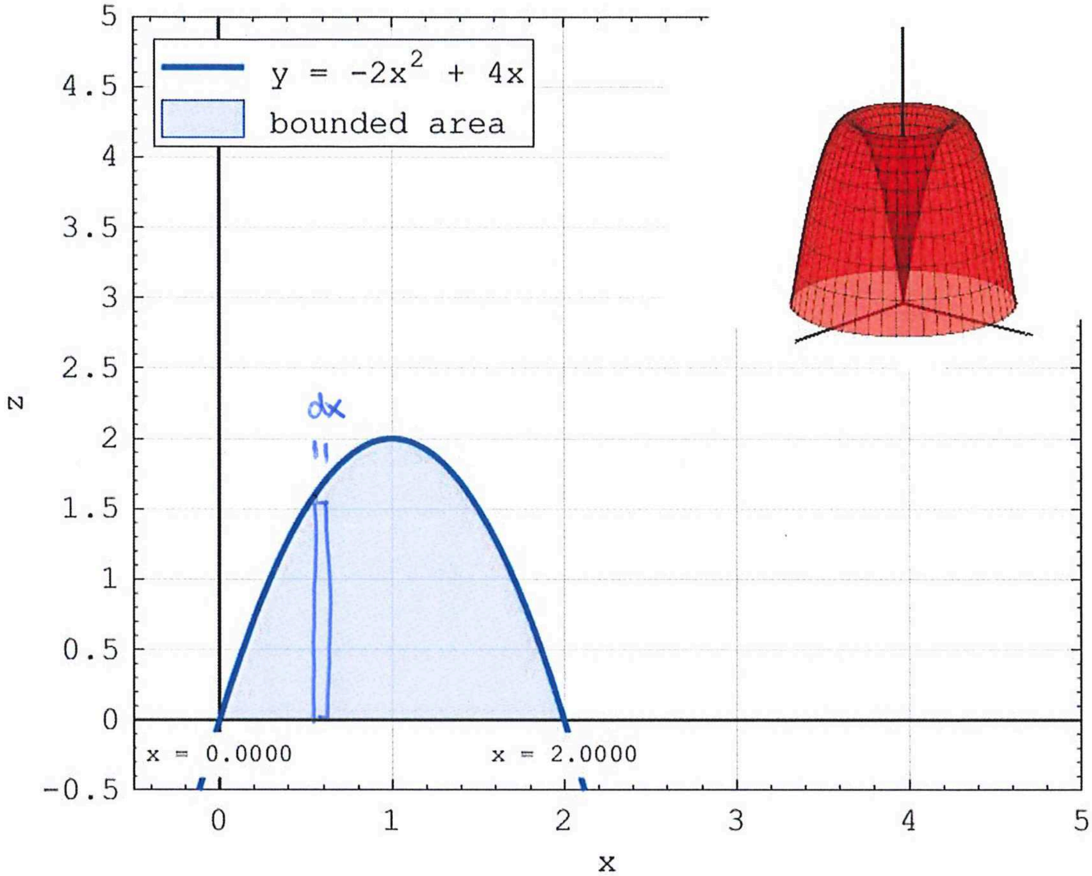
$$= \left[\frac{2^2}{2} - \frac{8}{3}\left(\frac{6-2}{4}\right)^{\frac{3}{2}} - 2\right] - \left[\frac{(-0.25)^2}{2} - \frac{8}{3}\left(\frac{6+0.25}{4}\right)^{\frac{3}{2}} + 0.25\right] + 2\left[-\frac{8}{3}\left(\frac{6-6}{4}\right)^{\frac{3}{2}} + \frac{8}{3}\left(\frac{6-2}{4}\right)^{\frac{3}{2}}\right]$$

$$= \left[\cancel{2} - \frac{8}{3} - \cancel{2}\right] - \left[\frac{1}{32} - 5.208 + 0.25\right] + 2\left[0 + \frac{8}{3}\right]$$

$$= -\frac{8}{3} - \frac{1}{32} + 5.208 - 0.25 + \frac{16}{3} = 7.59$$

$$\boxed{A = 7.59}$$

(5 marks) Calculate the volume of the bounded area of the function given in the legend (shaded in blue) revolved around the z-axis (as shown in the top left corner shaded in red).



shells:

$$d\forall = 2\pi r h \, dx = 2\pi x \left[-2x^2 + 4x\right] dx$$

$$\forall = 2\pi \int_0^2 -2x^3 + 4x^2 \, dx$$

$$= 2\pi \left[-\frac{1}{2}x^4 + \frac{4}{3}x^3\right]_0^2$$

$$= 2\pi \left[-\frac{1}{2}(2)^4 + \frac{4}{3}(2)^3\right] - 2\pi \left[\cancel{-\frac{1}{2}(0)^4}^{\,0} + \cancel{\frac{4}{3}(0)^3}^{\,0}\right]$$

$$= 2\pi \left[-8 + \frac{32}{3}\right]$$

$$\boxed{\forall = 16.76}$$

(5 marks) Calculate the volume of the bounded area of the function given in the legend (shaded in blue) revolved around the z-axis (as shown in the top left corner shaded in red).

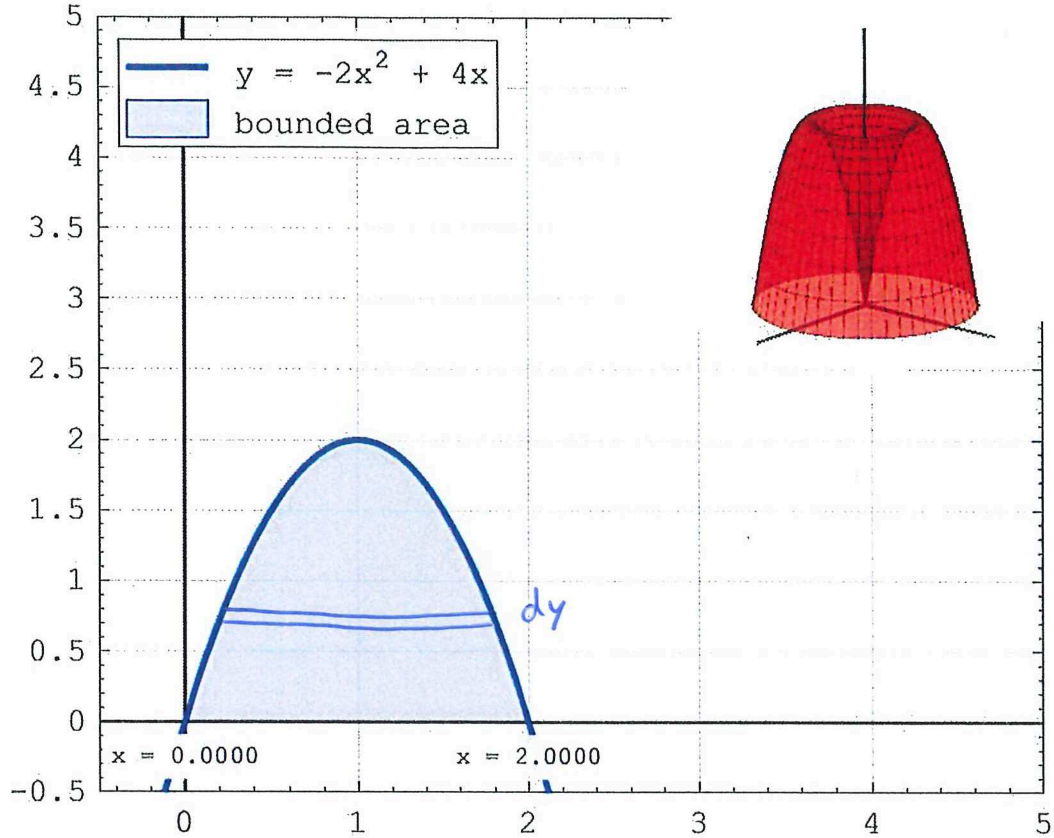
solve for x

$0 = -2x^2 + 4x - y$

$0 = 2x^2 - 4x + y$

$x = \frac{4 \pm \sqrt{16 - 4(2)(y)}}{2(2)}$

$= 1 \pm \frac{1}{4}\sqrt{16 - 8y}$



disks:

$$dV = \pi R^2 \, dy - \pi r^2 \, dy = \pi \int_0^2 \left(1 + \frac{1}{4}(16-8y)^{\frac{1}{2}}\right)^2 - \left(1 - \frac{1}{4}(16-8y)^{\frac{1}{2}}\right)^2 dy$$

$$V = \pi \int_0^2 \cancel{1} + \frac{1}{2}(16-8y)^{\frac{1}{2}} + \cancel{\frac{1}{16}(16-8y)} - \cancel{1} + \frac{1}{2}(16-8y)^{\frac{1}{2}} - \cancel{\frac{1}{16}(16-8y)} \, dy$$

$$= \pi \int_0^2 (16-8y)^{\frac{1}{2}} \, dy = \pi \left[-\frac{2}{24}(16-8y)^{\frac{3}{2}}\right]_0^2$$

$$= \pi \left[\cancelto{0}{-\frac{1}{12}(16-8(2))^{\frac{3}{2}}} + \frac{1}{12}(16-8(0))^{\frac{3}{2}}\right] = \frac{\pi}{12}(16)^{\frac{3}{2}} = \frac{\pi}{12}4^3 = \frac{4^2\pi}{3}$$

$$\boxed{V = \frac{4^2\pi}{3} = 16.76}$$