Rules of Differentiation:

Derivative of a constant:

$$\boxed{\frac{d}{dx}(c) = 0}$$

$$\frac{d}{dx}(c) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

Derivative of a constant times a function

$$\boxed{\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))}$$

$$\frac{d}{dx}(cf(x)) = \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} = c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = c\frac{d}{dx}(f(x))$$

Sum of two functions:

$$\boxed{\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))}$$

$$\frac{d}{dx}(f(x) + g(x)) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - f(x) - g(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

Multiplication of two functions (Product Rule)

$$\boxed{\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))}$$

$$\frac{d}{dx}(f(x)g(x)) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) + f(x+\Delta x)g(x) - f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)[g(x+\Delta x) - g(x)]}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x) \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

Divison of two functions (Quotient Rule)

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{g^2(x)}}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\,g(x)^{-1}\right) = f(x)\frac{d}{dx}\left(g(x)^{-1}\right) + g(x)^{-1}\frac{d}{dx}(f(x))$$

$$= f(x)\left[-g(x)^{-2}\frac{d}{dx}(g(x))\right] + g(x)^{-1}\frac{d}{dx}(f(x))\left[g(x)\,g(x)^{-1}\right]$$

$$= \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2} = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{g^2(x)}$$

Chain Rule:

$$\boxed{\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))}$$

$$\frac{d}{dx}(f(g(x))) = \lim_{\Delta x \to 0}\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

$$\frac{d}{dx}(g(x)) = \lim_{\Delta x \to 0}\frac{g(x+\Delta x) - g(x)}{\Delta x} \Rightarrow g(x+\Delta x) = g(x) + \frac{d}{dx}(g(x))\,\Delta x \qquad (1)$$

$$\frac{d}{dx}(f(x)) = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \Rightarrow f(x+\Delta x) = f(x) + \frac{d}{dx}(f(x))\,\Delta x \qquad (2)$$

$$f(g(x+\Delta x)) = f\Big(\underbrace{g(x)}_{\text{"x" in (2)}} + \underbrace{\frac{d}{dx}(g(x))\,\Delta x}_{\text{"}\Delta x\text{" in equ. (2)}}\Big) \qquad \text{substitution of equ. (1)}$$

$$f(g(x+\Delta x)) = f(g(x)) + \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x))\,\Delta x \qquad \text{substitution of equ. (2).}$$

so

$$\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = \frac{d}{dx}(f(g(x)))\frac{d}{dx}(g(x)) \qquad \text{rearrange.}$$

definition of $\frac{d}{dx}(f(g(x)))$